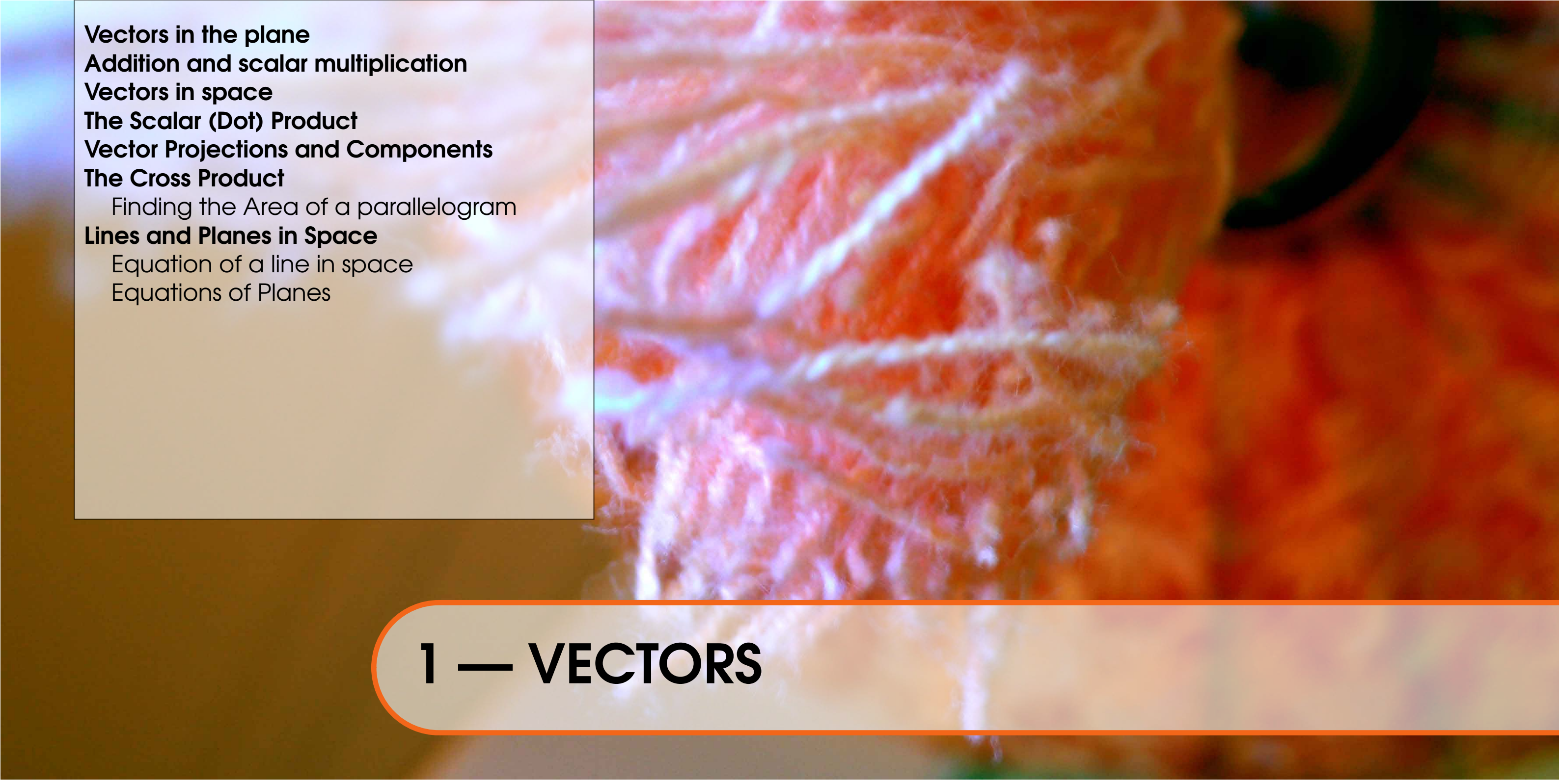




# 1 — VECTORS

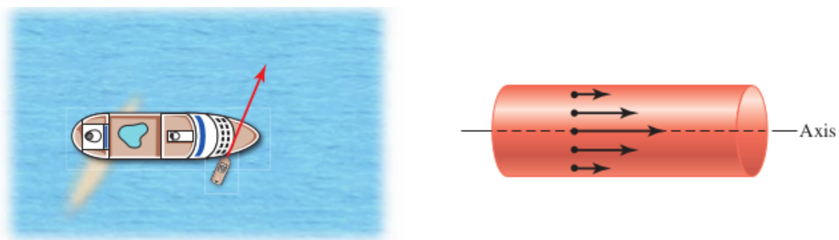
## 1.1 Vectors in the plane

**Definition 1.1.1** A scalar is a quantity that has magnitude but no direction.
A vector is usually described as a quantity that has both magnitude and direction.

Geometrically, a vector is represented by a directed line segment that is an arrow and is written either as a boldface symbol **v** or $\overrightarrow{v}$ or $\overrightarrow{AB}$ for instance weight, velocity, frictional force are vector quantity. The arrow points in the direction of the vector, and the length of the arrow gives the magnitude of the vector.

### Notations and Terminologies

A vector whose initial point is A and whose terminal point is B is given by $\overrightarrow{AB}$ and the magnitude (or length) of vector $\overrightarrow{AB}$ is denoted by $\|\overrightarrow{AB}\|$.
Figure 1.1a gives an aerial view of a tugboat trying to free a cruise liner that has run aground in shallow waters. The magnitude and direction of the force exerted by the tugboat are represented by the vector shown in the figure.
In Figure 1.1b the vectors (arrows) give the magnitude and direction of blood cells flowing through




(**a**) The vector represents the force exerted by a tugboat on a ship.

(**b**) The vectors represent the velocity of blood cells flowing through an artery.

Figure 1.1:

an artery. Observe that the lengths of the vectors vary; this reflects the fact that the blood cells near the central axis have a greater velocity than those near the walls of the artery.

Two vectors, **v** and **w** , that have the same magnitude and direction are said to be equal, written

$\mathbf{v} = \mathbf{w}$. Thus, the vectors $\mathbf{v} = \overrightarrow{AB}$ and $\mathbf{w} = \overrightarrow{CD}$. Graphically, this means that the arrows representing $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel, have the same length and are pointing in the same direction.

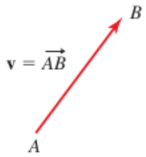



Figure 1.2: **v** is the directed line segment from A to B.

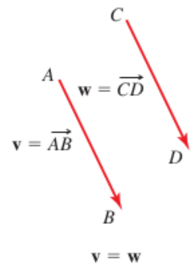



Figure 1.3: **v** and **w** have the same length and direction.

## 1.2 Addition and scalar multiplication

Consider the vector $\overrightarrow{AB}$ as representing the displacement of a particle from the point A to the point B, notice that the end result of displacing the particle from A to B (corresponding to the vector $\overrightarrow{AB}$), followed by displacing the particle from B to C (corresponding to the vector $\overrightarrow{BC}$) is the same as displacing the particle directly from A to C, which corresponds to the vector $\overrightarrow{AC}$ (called the resultant vector). We call AC the sum of AB and BC and write

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}.$$

To add two vectors, we locate the initial point of one at the terminal point of the other and complete the parallelogram, as indicated in Figure 1.5 . The vector lying along the diagonal, with initial point at A and terminal point at C is the sum

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD}.$$

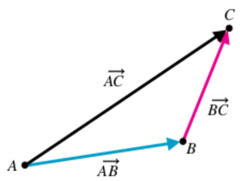



Figure 1.4: Resultant vector

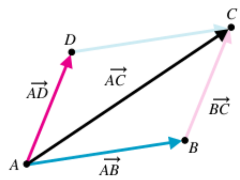



Figure 1.5: Sum of two vectors.

A vector can be multiplied by a scalar. If $c \neq 0$ is a scalar and **v** is a vector, then the scalar multiple of c and **v** is a vector c**v**. The magnitude of cv is $|c|\|\mathbf{v}\|$

Two nonzero vectors are parallel if they are scalar multiples of one another. In figure 1.5 vector $\overrightarrow{AD}$ and $\overrightarrow{BC}$ are parallel.

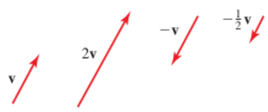



Figure 1.6: Scalar multiples of **v**

The difference of two vectors $\mathbf{v}$ and $\mathbf{w}$, written $\mathbf{v}-\mathbf{w}$, is defined by

$$\mathbf{v}-\mathbf{w}=\mathbf{v}+(-\mathbf{w})$$

To describe this operation geometrically, consider once again the two vectors $\mathbf{v}$ and $\mathbf{w}$ of Figure 1.7, which are reproduced in Figure 1.8. If we translate $\mathbf{w}$ , reverse it to obtain $-\mathbf{w}$, and then use the parallelogram law to add $\mathbf{v}$ to $-\mathbf{w}$ , we obtain $\mathbf{v}-\mathbf{w}$ , as shown in Figure 1.8


Figure 1.7: The vectors $\mathbf{v}$ and $\mathbf{w}$

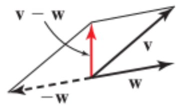



Figure 1.8: The vector $\mathbf{v}-\mathbf{w}$.

### Vectors in the Coordinate Plane

The vector $\mathbf{v}$ with initial point at the origin and terminal point $P(v_1,v_2)$ is called the **position vector** of the point $P(v_1,v_2)$ and is denoted by $\langle v_1,\ v_2\rangle$.

**Definition 1.2.1** A vector in the plane is an ordered pair $v=\langle v_1,\ v_2\rangle$ of real numbers, $v_1$ and $v_2$ , called the **scalar components** of $\mathbf{v}$. The **zero vector** is $0=\langle 0,\ 0\rangle$.

**Definition 1.2.2** Given the points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ , the vector $\overrightarrow{P_1P_2}$ is represented by the position vector

$$\mathbf{v}=\overrightarrow{P_1P_2}=\langle x_2-x_1,\ y_2-y_1\rangle \tag{1.1}$$

**Example 1.1** Find the vector $\mathbf{v}$ with initial point $A(-3,-2)$ and terminal point $B(2,1)$ .

**Solution:** Using Equation 1.1, we find the vector $\mathbf{v}$ to be

$$v=\langle 2-(-3),\ 1-(-1)\rangle=\langle 5,\ 2\rangle$$

**Example 1.2** Let $\mathbf{v}$ be a vector with initial point $A(0,0)$ and terminal point $B(3,2)$ , and let $\mathbf{u}$ be a vector with initial point $C(1,3)$ and terminal point $D(4,5)$. Show that $\mathbf{v}=\mathbf{u}$.

**Solution:** To show that $\mathbf{v}=\mathbf{u}$, we need to show that both vectors have the same length and direction. Using the distance formula, we find

$$\text{length of } \overrightarrow{AB}=\sqrt{(3-0)^2+(2-0)^2=\sqrt{13}}$$

and

$$\text{length of } \overrightarrow{CD}=\sqrt{(4-1)^2+(5-3)^2=\sqrt{13}}$$

so $\mathbf{v}$ and $\mathbf{u}$ have the same length. Next, we find

$$\text{slope of } \overrightarrow{AB}=\frac{2-0}{3-0}=\frac{2}{3}$$

and

$$\text{slope of } \overrightarrow{CD}=\frac{5-3}{4-1}=\frac{2}{3}$$

so $\mathbf{v}$ and $\mathbf{u}$ have the same direction. This proves that $\mathbf{v}=\mathbf{u}$

### Length of a Vector

**Definition 1.2.3** The length or magnitude of $\mathbf{v} = \langle v_1,\ v_2 \rangle$ is

$$\|v\| = \sqrt{v_1^2 + v_2^2} \tag{1.2}$$

### Vector Addition in the Coordinate Plane

**Parallelogram Law for Vector Addition**
If $\mathbf{u} = \langle u_1,\ u_2 \rangle$ and $\mathbf{v} = \langle v_1,\ v_2 \rangle$ , then

$$\mathbf{u} + \mathbf{v} = \langle u_1,\ u_2 \rangle + \langle v_1,\ v_2 \rangle = \langle u_1 + v_1,\ u_2 + v_2 \rangle$$

■ **Example 1.3** If $\mathbf{u} = \langle 3,\ -2 \rangle$ and $\mathbf{v} = \langle -1,\ 3 \rangle$ , then

$$\mathbf{u} + \mathbf{v} = \langle 3 + (-1),\ -2 + 3 \rangle = \langle 2,\ 1 \rangle$$ ■

■ **Example 1.4** Suppose ship A is cursing in the easterly direction at a speed of 15km/hr while ship B is speeding at 30km/hr in the north east direction. Assuming that both ships started from the same place at the same time, find the rate at which the two ships are separating. ■

**solution:** Taking the x-axis the easterly direction and y-axis the northerly direction see figure 1.9, we see that the velocity of the ship A is represented by the vector $A = 15\,i$ and the velocity of the ship B by the vector

$$B = 30\left(\cos\frac{\pi}{4}\,i + \sin\frac{\pi}{4}\,j\right)$$

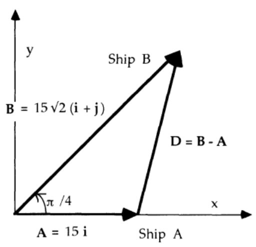



Figure 1.9: Rate of displacement between two ships

The displacement of the ship B relative to the ship A is represented by the vector $D = B - A$. Hence the rate at which the ships separating is equal to the magnitude of the displacement vector $D$. Thus,

$$\begin{aligned} D &= B - A = 15(\sqrt{2} - 1)\,i + 15\sqrt{2}\,j \\ \Longrightarrow \quad \|D\| &= 15\sqrt{(\sqrt{2} - 1)^2 + 2} = 22.1 km/hr \end{aligned}$$

**Scalar Multiplication**
If $\mathbf{u} = \langle u_1,\ u_2 \rangle$ and $c$ is a scalar, then,

$$c\mathbf{u} = \langle cu_1,\ cu_2 \rangle$$

### Properties of Vectors

**Theorem 1.2.1 — Rules for Vector Addition and Scalar Multiplication.** Suppose that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors and that $c$ and $d$ are scalars. Then

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = 0$
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
6. $c(d\mathbf{u}) = (cd)\mathbf{u}$
7. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
8. $1\mathbf{u} = \mathbf{u}$

### Unit Vectors

A unit vector is a vector of length 1. Unit vectors are primarily used as indicators of direction. For example, if $\mathbf{w}$ is a nonzero vector, then the vector

$$\mathbf{u} = \frac{\mathbf{w}}{\|\mathbf{w}\|}$$

is a unit vector having the same direction as $\mathbf{w}$. Furthermore, by writing $\mathbf{w}$ in the form

$$\mathbf{w} = \|\mathbf{w}\|\left(\frac{\mathbf{w}}{\|\mathbf{w}\|}\right) = \|\mathbf{w}\|\mathbf{u}$$

the two properties of magnitude and direction that define a vector are clearly displayed.

**Example 1.5** Find a unit vector in the same direction as $\mathbf{v} = \langle 4, 3\rangle$

**Solution:** $\|\mathbf{v}\| = \sqrt{4^2+3^2} = \sqrt{25} = 5$

The unit vector in the same direction as $\mathbf{v} = \langle 4, 3\rangle$ is then

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle 4,3\rangle}{5} = \left\langle \frac{4}{5}, \frac{3}{5}\right\rangle$$

### Standard Basis Vectors

There are two unit vectors in the coordinate plane that are singled out for a special role. They are the vectors i and j defined by

$$i = \langle 1, 0\rangle \quad \text{and} \quad j = \langle 0, 1\rangle$$

The vector $i$ points in the positive x-direction, whereas the vector $j$ points in the positive y-direction. Let $\mathbf{u} = \langle u_1,\ u_2\rangle$ be a vector in the coordinate plane. Then

$$\begin{aligned} \mathbf{u} &= \langle u_1,\ u_2\rangle = \langle u_1,\ 0\rangle + \langle 0,\ u_2\rangle \\ &= u_1\langle 1,\ 0\rangle + u_2\langle 0,\ 1\rangle \\ &= u_1\, i + u_2\, j \end{aligned}$$

This shows that any vector in the plane can be expressed in terms of the vectors $i$ and $j$. For this reason the vectors $i$ and $j$ are referred to as standard basis vectors.

**Example 1.6** Let $\mathbf{u} = -4i+2j,\quad \mathbf{v} = 2i+j,\quad$ and $\mathbf{w} = 2i+3j$. Find the scalar $m$ and $n$ such that $\mathbf{w} = m\mathbf{u}+n\mathbf{v}$

**Solution:** We have

$$\begin{aligned} m\mathbf{u}+n\mathbf{v} &= m(-4i+2j)+n(2i+j) \\ &= (-4m+2n)\,i+(2m+n)\,j = 2i+3j \end{aligned}$$

Equating the corresponding components, we obtain

$$-4m+2n = 2 \text{ and } 2m+n = 3$$

Solving for $m$ and $n$, we get $m = \frac{1}{2}$ and $n = 2$

**Example 1.7** At a certain point during a jump, there are two principal forces acting on a sky diver: gravity exerting a force of 180 pounds straight down and air resistance exerting a force of 180 pounds up and 30 pounds to the right. What is the net force acting on the sky diver?

**Solution:** We write the gravity force vector as $\mathbf{g} = \langle 0, -180\rangle$ and the air resistance force vector as $\mathbf{r} = \langle 30, 180\rangle$. The net force on the sky diver is the sum of the two forces, $\mathbf{g}+\mathbf{r} = \langle 30, 0\rangle$. We illustrate the forces in Figure 1.10. Notice that at this point, the vertical forces are balanced, producing a "free-fall" vertically, so that the sky diver is neither accelerating nor decelerating vertically. The net force is purely horizontal, combating the horizontal motion of the sky diver after jumping from the plane.

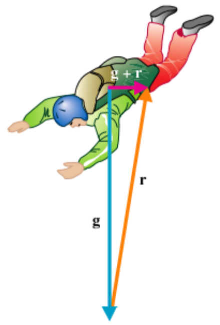



Figure 1.10: Forces on a sky diver.

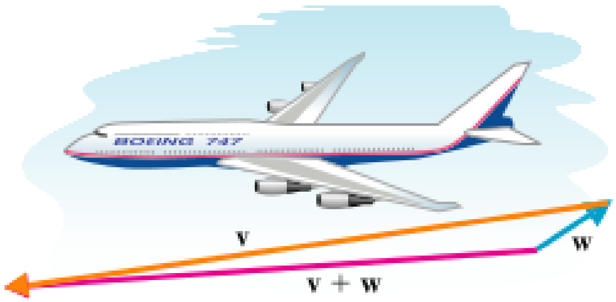



Figure 1.11: Forces on an airplane.

■ **Example 1.8** An airplane has an airspeed of 400 mph. Suppose that the wind velocity is given by the vector $\mathbf{w} = \langle 20, 30\rangle$. In what direction should the airplane head in order to fly due west i.e., in the direction of the unit vector $-i = \langle -1, 0\rangle$? ■

**Solution:** We illustrate the velocity vectors for the airplane and the wind in Figure 1.11. We let the airplane's velocity vector be $\mathbf{v} = \langle x, y\rangle$. The effective velocity of the plane is then $\mathbf{v}+\mathbf{w}$, which we set equal to $\langle c, 0\rangle$, for some negative constant c. Since

$$\mathbf{v}+\mathbf{w} = \langle x+20, y+30\rangle = \langle c, 0\rangle$$

we must have $x+20 = c$ and $y+30 = 0$, so that $y = -30$. Further, since the plane's airspeed is 400 mph we must have $400 = \|\mathbf{v}\| = \sqrt{x^2+y^2} = \sqrt{x^2+900}$. Squaring this gives us $x^2+900 = 160,000$, so that $x = -159,100$. (We take the negative square root so that the plane heads westward.) Consequently, the plane should head in the direction of $\mathbf{v} = \langle -159,100,-30\rangle$ which points left and down, or southwest, at an angle of $tan^{-1}(30/159,100) \approx 4^o$ below due west.

■ **Example 1.9** A body weighing 100 kg is suspended from two ropes as shown in figure 1.12. Find the tension in the ropes. ■

**Solution:** Let the tension in the ropes be denoted by $F_1 = a_1 i + a_2 j$ and $F_2 = b_1 i + b_2 j$. The weight of the body is represented by the vector $W = -100\, j$. Since the system is in equilibrium the sum of all the forces must be equal to zero. Thus,

$$F_1 + F_2 + W = (a_1+b_1)\, i + (a_2+b_2-100)\, j = 0.$$ This implies

$$a_1+b_1 = 0, \quad a_2+b_2-100 = 0$$

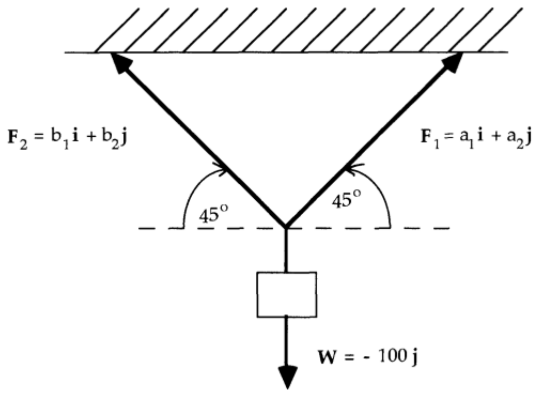



Figure 1.12: Tension in ropes from which a weight is suspended.

Hence, $a_1 = -b_1$ and $a_2 + b_2 = 100$. By symmetry, we know that $\|F_1\| = \|F_2\|$, which implies that $a_2 = b_2$. Therefore, $a_2 = b_2 = 50$. Since $a_1 = \|F_1\|\cos 45$ and $a_2 = \|F_2\|\sin 45$, it follows that $\|F_1\| = \|F_2\| = 50\sqrt{2}$ and $a_1 = -b_1 = 50$. Thus, $F_1 = 50(i+j)$ and $F_2 = 50(-i+j)$. Notice that $F_1 + F_2 = 100\,j$

## 1.3 Vectors in space

A vector in 3-space is an ordered triple of real numbers

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle,$$

where $a_1, a_2$, and $a_3$ are the components of the vector. In particular, the position vector of a point $P(x_1, y_1, z_1)$ is the vector We represent the vector $\langle a_1, a_2, a_3 \rangle$ as an arrow from the origin $(0,0,0)$ to the point $(a_1, a_2, a_3)$ in 3-space. In this way, the direction indicated by the arrow, as viewed from the origin, gives the direction of the vector.

**Definition 1.3.1** If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ are vectors in 3-space and c is a scalar, then

1. $\mathbf{a} = \mathbf{b}$ if and only if $a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3$
2. $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$
3. $c\mathbf{a} = \langle ca_1, ca_2, ca_3 \rangle$
4. $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ (Length or norm of a vector)

The vector with initial point $P_1(x_1, y_1, z_1)$ and terminal point $P_2(x_2, y_2, z_2)$ is

$$\overrightarrow{P_1P_2} = \langle x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1 \rangle$$

Thus, we can find the components of a vector by subtracting the respective coordinates of its initial point from the coordinates of its terminal point, as illustrated in Figure 1.13. The vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ are the position vectors of the point $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$. As a natural extension of the definition of vector subtraction in 2-space into 3-space, we have

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = \langle x_2,\ y_2,\ z_2 \rangle - \langle x_1,\ y_1,\ z_1 \rangle = \langle x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1 \rangle$$

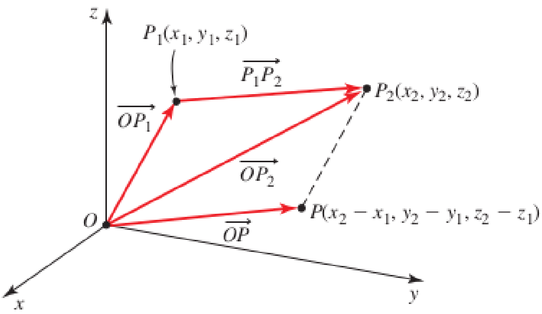



Figure 1.13:

By considering the parallelogram $OPP_2P_1$ in Figure 1.13, you can convince yourself that $\overrightarrow{P_1P_2}$ is represented by the position vector $\overrightarrow{OP}$ of the point $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

■ **Example 1.10** Let $P(2, -1, 2)$ and $Q(1, 4, 5)$ be two points in 3-space.

(a) Find the vector $\overrightarrow{PQ}$.

(b) Find $\|\overrightarrow{PQ}\|$.

(c) Find a unit vector having the same direction as $\overrightarrow{PQ}$

■

**Solution:**

(a) $\overrightarrow{PQ} = \langle 1-2,\ 4-(-1),\ 5-2\rangle = \langle -1,\ 5,\ 3\rangle$

(b) $\|\overrightarrow{PQ}\| = \sqrt{(-1)^2+5^2+3^2} = \sqrt{35}$

(c) Using the results of parts (a) and (b), we obtain the unit vector

$$\mathbf{u} = \frac{\overrightarrow{PQ}}{\|\overrightarrow{PQ}\|} = \frac{\langle -1,\ 5,\ 3\rangle}{\sqrt{35}} = \left\langle -\frac{1}{\sqrt{35}},\ \frac{5}{\sqrt{35}},\ \frac{3}{\sqrt{35}}\right\rangle$$

### Standard Basis Vectors in Space

In three-dimensional space, the 3-space vectors

$$i = \langle 1,0,0\rangle, \quad j = \langle 0,1,0\rangle \quad \text{and} \quad k = \langle 0,0,1\rangle$$

form a basis for the space, in the sense that any vector in the space can be expressed in terms of these vectors. If $\mathbf{a} = \langle a_1,\ a_2,\ a_3\rangle$ be a vector in space. Then

$$\begin{aligned}\mathbf{a} &= \langle a_1,\ a_2,\ a_3\rangle = \langle a_1,\ 0,\ 0\rangle + \langle 0,\ a_2,\ 0\rangle + \langle 0,\ 0,\ a_3\rangle \\ &= a_1\langle 1,\ 0,\ 0\rangle + a_2\langle 0,\ 1,0\rangle + a_3\langle 0,\ 0,1\rangle \\ &= a_1 i + a_2 j + a_3 k\end{aligned}$$

## 1.4 The Scalar (Dot) Product

**Definition 1.4.1** Let $\mathbf{a} = \langle a_1,\ a_2,\ a_3\rangle$ and $\mathbf{b} = \langle b_1,\ b_2,\ b_3\rangle$ be any two vector in space. Then the dot product of **a** and **b** is the number $\mathbf{a}.\mathbf{b}$ defined by

$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

**Example 1.11** Find the dot product of each pair of vectors:

1. $v = \langle 2,-3\rangle$ and $u = \langle 3,1\rangle$
2. $v = \langle 1,-2,4\rangle$ and $u = \langle 1,1,0\rangle$

**Properties of the Dot Product**

Let $\mathbf{u}, \mathbf{v},$ and $\mathbf{w}$ be vectors in 2- or 3-space and let $c$ be a scalar. Then

1. $\mathbf{u}.\mathbf{v} = \mathbf{v}.\mathbf{u}$
2. $\mathbf{u}.(\mathbf{v}+\mathbf{w}) = \mathbf{u}.\mathbf{v} + \mathbf{u}.\mathbf{w}$
3. $(c\mathbf{u}).\mathbf{v} = c(\mathbf{u}.\mathbf{v}) = \mathbf{u}.(c\mathbf{v})$
4. $\mathbf{u}.\mathbf{u} = \|\mathbf{u}\|^2$
5. $\mathbf{0}.\mathbf{u} = \mathbf{0}$

### The Angle Between Two Vectors

The **angle between two nonzero vectors** is the angle $\theta$ between their corresponding position vectors, where $0 \le \theta \le \pi$.

R If two vectors are parallel, then $\theta = 0$ or $\theta = \pi$
The angle between the zero vector and another vector is not defined.

**Theorem 1.4.1** Let $\theta$ be the angle between two nonzero vectors **a** and **b** . Then

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}$$

*Proof.* Consider the triangle determined by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$ as shown in Figure 1.14. Using the law of cosines, we have

$$\|\mathbf{a}-\mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\,\|\mathbf{b}\|\cos\theta$$

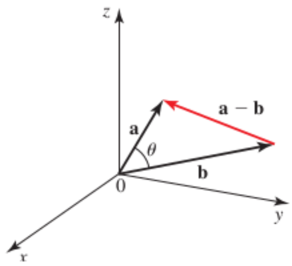



Figure 1.14: The angle between **a** and **b** is $\theta$.

But

$$\begin{aligned} \|\mathbf{a}-\mathbf{b}\|^2 &= (\mathbf{a}-\mathbf{b}).(\mathbf{a}-\mathbf{b}) \\ &= \mathbf{a}.\mathbf{a}-\mathbf{a}.\mathbf{b}-\mathbf{b}.\mathbf{a}+\mathbf{b}.\mathbf{b} \\ &= \|\mathbf{a}\|^2-2\mathbf{a}.\mathbf{b}+\|\mathbf{b}\|^2 \end{aligned}$$

so we have

$$\begin{aligned} \|\mathbf{a}\|^2-2\mathbf{a}.\mathbf{b}+\|\mathbf{b}\|^2 &= \|\mathbf{a}\|^2+\|\mathbf{b}\|^2-2\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta \\ \Longrightarrow \quad -2\mathbf{a}.\mathbf{b} &= -2\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta \\ \Longrightarrow \quad \cos\theta &= \frac{\mathbf{a}.\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} \end{aligned}$$

■

■ **Example 1.12** If the vectors **a** and **b** have lengths 4 and 6, and the angle between them is $\frac{\pi}{3}$, find **a**.**b**. ■

**Solution:** We have

$$\mathbf{a}.\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta = 4.6.\cos\frac{\pi}{3} = 24.\frac{1}{2} = 12$$

■ **Example 1.13** Find the angle between the vectors $\mathbf{a} = \langle 2,-1,-2\rangle$ and $\mathbf{b} = \langle 1,2,2\rangle$ ■

**Solution:** We have

$\|\mathbf{a}\| = \sqrt{2^2+(-1)^2+(-2)^2} = 3$, $\|\mathbf{b}\| = \sqrt{1^2+2^2+2^2} = 3$ and $\mathbf{a}.\mathbf{b} = 2(1)+(-1)2+(-2)2 = -4$

Hence, $\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \frac{-4}{9} \quad \Longrightarrow \quad \theta = \cos^{-1}\left(\frac{-4}{9}\right) =$

**Theorem 1.4.2** Two vectors **a** and **b** are orthogonal if and only if $\mathbf{a}.\mathbf{b} = 0$.

■ **Example 1.14** Determine whether the following pairs of vectors are orthogonal:

(a) $\mathbf{a} = -i+2j+5k, \quad \mathbf{b} = 3i+4j-k$

(b) $\mathbf{u} = i-j+2k, \quad \mathbf{v} = 2i-j+k$

■

**Solution:**

(a) Since $\mathbf{a}.\mathbf{b} = (-1)3+2(4)+5(-1) = 0$. So, **a** and **b** are orthogonal.

(b) Since $\mathbf{u}.\mathbf{v} = 1(2)+(-1)(-1)+2(1) = 5 \neq 0$. So, **u** and **v** are not orthogonal.

**Theorem 1.4.3 — Cauchy-Schwartz Inequality.** For any vectors **a** and **b**,

$$|\mathbf{a}.\mathbf{b}| \leq \|\mathbf{a}\|\|\mathbf{b}\|$$

**Theorem 1.4.4 — The Triangle Inequality.** For any vectors **a** and **b**,

$$\|\mathbf{a}+\mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$$

### Direction Cosines

We can describe the direction of a nonzero vector **a** by giving the angles $\alpha$, $\beta$, and $\gamma$ that a makes with the positive x -, y -, and z -axes, respectively. (See Figure 1.15.) These angles are called the direction angles of **a**. The cosines of these angles, $\cos\alpha$, $\cos\beta$, and $\cos\gamma$, are called the direction cosines of the vector **a**.

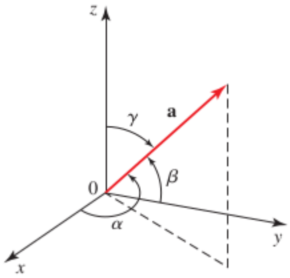



Figure 1.15: The direction angles of a vector

Let $\mathbf{a} = a_1\,i + a_2\,j + a_3\,k$ be a nonzero vector in 3-space. Then

$$\mathbf{a}.i = (a_1\,i + a_2\,j + a_3\,k).i = a_1$$

So,

$$\cos\alpha = \frac{\mathbf{a}.i}{\|\mathbf{a}\|\|i\|} = \frac{a_1}{\|\mathbf{a}\|}$$

Similarly,

$$\cos\beta = \frac{\mathbf{a}.j}{\|\mathbf{a}\|\|j\|} = \frac{a_2}{\|\mathbf{a}\|}, \quad \text{and} \quad \cos\gamma = \frac{\mathbf{a}.k}{\|\mathbf{a}\|\|k\|} = \frac{a_3}{\|\mathbf{a}\|}$$

By squaring and adding the three direction cosines, we obtain

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{a_1^2}{\|\mathbf{a}\|^2} + \frac{a_2^2}{\|\mathbf{a}\|^2} + \frac{a_3^2}{\|\mathbf{a}\|^2} = \frac{\|\mathbf{a}\|^2}{\|\mathbf{a}\|^2} = 1$$

R

1. If $\mathbf{a} = a_1\,i + a_2\,j + a_3\,k$ is nonzero, then the unit vector having the same direction as **a** is

$$\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{a_1}{\|\mathbf{a}\|}i + \frac{a_2}{\|\mathbf{a}\|}j + \frac{a_3}{\|\mathbf{a}\|}k$$
$$= (\cos\alpha)\mathbf{i} + (\cos\beta)\mathbf{j} + (\cos\gamma)\mathbf{k} \quad (1.3)$$

   This shows that the direction cosines of **a** are the components of the unit vector in the direction of **a**.
2. From Equation (1.3) we see that

$$\mathbf{a} = \|\mathbf{a}\|\left[(\cos\alpha)\mathbf{i} + (\cos\beta)\mathbf{j} + (\cos\gamma)\mathbf{k}\right]$$

■ **Example 1.15** Find the direction angles of the vector $\mathbf{a} = 2i + j + 2k$ ■

**Solution:** We have $\|\mathbf{a}\| = \sqrt{2^2+1^2+2^2} = 3$. So,

$$\cos\alpha = \frac{2}{3}, \quad \cos\beta = \frac{1}{3}, \quad \text{and } \cos\gamma = \frac{2}{3}$$

Therefore,

$$\alpha = \cos^{-1}\left(\frac{2}{3}\right) \approx 48.19^o, \quad \beta = \cos^{-1}\left(\frac{1}{3}\right) \approx 70.53^o, \quad \text{and } \gamma = \cos^{-1}\left(\frac{2}{3}\right) \approx 48.19^o$$

## 1.5 Vector Projections and Components

Figure 1.16a depicts a child pulling a sled with a constant force represented by the vector **F**. The force **F** can be expressed as the sum of two forces: a horizontal component $\mathbf{F_1}$ and a vertical component $\mathbf{F_2}$, as shown in Figure 1.16b.

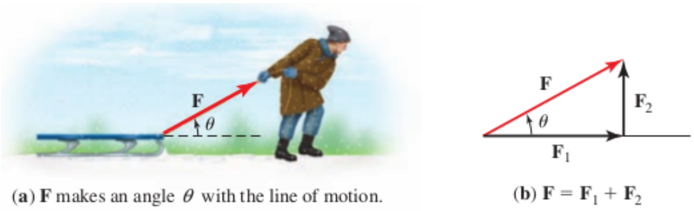



Figure 1.16:

Observe that $\mathbf{F_1}$ acts in the direction of motion, whereas $\mathbf{F_2}$ acts in a direction perpendicular to the direction of motion.

More generally, we are interested in the component of one vector **b** in the direction of another nonzero vector **a**. The vector that is obtained by projecting **b** onto the line containing the vector **a** is called the vector projection of **b** onto **a** (also called the vector component of **b** along **a**) and denoted by

$$\text{proj}_{\mathbf{a}}^{\mathbf{b}}$$

(See Figure 1.17.) The scalar projection of **b** onto **a** (also called the scalar component of **b** along **a**) is

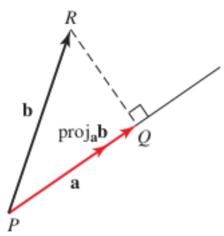



Figure 1.17: $\text{proj}_{\mathbf{a}}^{\mathbf{b}}$ : $(\vec{PQ})$ is the vector projection of **b** onto **a**.

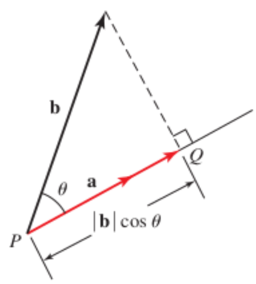



Figure 1.18: $\text{comp}_{\mathbf{a}}^{\mathbf{b}} = \|b\| \cos\theta$.

the length of $\text{proj}_{\mathbf{a}}^{\mathbf{b}}$ if the projection has the same direction as **a** and the negative of the length of $\text{proj}_{\mathbf{a}}^{\mathbf{b}}$ if the projection has the opposite direction. We denote this scalar projection by

$$\text{comp}_{\mathbf{a}}^{\mathbf{b}}$$

Since

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}$$

we can write

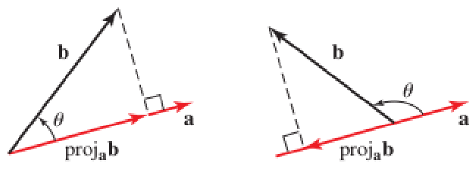
$$\|\mathbf{b}\| \cos\theta = \frac{|\mathbf{b}\|(\mathbf{a}.\mathbf{b})}{\|\mathbf{a}\|\|\mathbf{b}\|} = \frac{\mathbf{a}.\mathbf{b}}{\|\mathbf{a}\|}$$

Therefore, the component of **b** along **a** is

$$\text{comp}_{\mathbf{a}}^{\mathbf{b}} = \frac{\mathbf{b}.\mathbf{a}}{\|\mathbf{a}\|}$$

The vector projection of **b** along **a** is the component of **b** along **a** times the direction of **a**.

$$\text{proj}_{\mathbf{a}}^{\mathbf{b}} = \left(\frac{\mathbf{b}.\mathbf{a}}{\|\mathbf{a}\|}\right) \frac{\mathbf{a}}{\|\mathbf{a}\|} = \left(\frac{\mathbf{a}.\mathbf{b}}{\|\mathbf{a}\|^2}\right) \mathbf{a}$$




(a) θ is acute. (b) θ is obtuse.

Figure 1.19: $\text{proj}_{\mathbf{a}}^{\mathbf{b}}$ points in the same direction as **a** if $\theta$ is acute and points in the opposite direction as **a** if $\theta$ is obtuse.

**Example 1.16** Let $\mathbf{a} = 2i + 3j - 4k$ and $\mathbf{b} = 3i - 2j + k$. Find
(a) the component of **b** along **a** (b) $\text{proj}_{\mathbf{a}}^{\mathbf{b}}$ (c) $\text{proj}_{\mathbf{b}}^{\mathbf{a}}$ ■

**Solution:** (a) The scalar component of **b** along **a** is

$$\text{comp}_{\mathbf{a}}^{\mathbf{b}} = \frac{\mathbf{a}.\mathbf{b}}{\|\mathbf{a}\|} = \frac{2(3)+3(-2)+(-4)1}{\sqrt{2^2+3^2+(-4)^2}} = -\frac{4}{\sqrt{29}}$$

(b) The projection of vector **b** along **a** is

$$\text{proj}_{\mathbf{a}}^{\mathbf{b}} = \left(\frac{\mathbf{a}.\mathbf{b}}{\|\mathbf{a}\|^2}\right)\mathbf{a} = -\frac{4}{29}(2i+3j-4k) = -\frac{8}{29}i - \frac{12}{29}j + \frac{16}{29}k$$

(c) The projection of vector **a** along **b** is

$$\text{proj}_{\mathbf{b}}^{\mathbf{a}} = \left(\frac{\mathbf{a}.\mathbf{b}}{\|\mathbf{b}\|^2}\right)\mathbf{b} = -\frac{4}{3^2+(-2)^2+1^2}(3i-2j+k) = -\frac{6}{7}i + \frac{4}{7}j + -\frac{2}{7}k$$

Using vector projections, we can express any vector **b** as the sum of a vector parallel to a vector **a** and a vector perpendicular to **a**. In fact, from Figure 1.20 we see that.

$$\begin{aligned}\mathbf{b} &= \text{proj}_{\mathbf{a}}^{\mathbf{b}} + (\mathbf{b} - \text{proj}_{\mathbf{a}}^{\mathbf{b}}) \\ &= \left(\frac{\mathbf{a}.\mathbf{b}}{\|\mathbf{a}\|^2}\right)\mathbf{a} + \left[\mathbf{b} - \left(\frac{\mathbf{a}.\mathbf{b}}{\|\mathbf{a}\|^2}\right)\mathbf{a}\right]\end{aligned}$$

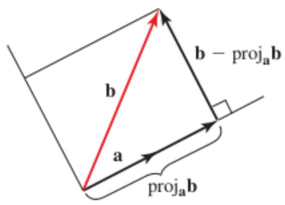



Figure 1.20: The direction angles of a vector

**Example 1.17** Write $\mathbf{b} = 3i - j + 2k$ as the sum of a vector parallel to $\mathbf{a} = 2i - j + k$ and a vector perpendicular to **a**. ■

**Solution:** We have

$$\mathbf{a}.\mathbf{b} = 2(3) + (-1)(-1) + 1(2) = 9 \text{ and } \|\mathbf{a}\|^2 = \mathbf{a}.\mathbf{a} = 2^2 + (-1)^2 + 1^2 = 6$$

Hence,

$$\begin{aligned}\mathbf{b} &= \left(\frac{\mathbf{a}.\mathbf{b}}{\|\mathbf{a}\|^2}\right)\mathbf{a} + \left[\mathbf{b} - \left(\frac{\mathbf{a}.\mathbf{b}}{\|\mathbf{a}\|^2}\right)\mathbf{a}\right] \\ &= \frac{9}{6}(2i-j+k) + \left[(3i-j+2k) - \frac{9}{6}(2i-j+k)\right] \\ &= \left(3i - \frac{3}{2}j + \frac{3}{2}k\right) + \left(\frac{1}{2}j + \frac{1}{2}k\right)\end{aligned}$$

Therefor, the vector $\left(3i - \frac{3}{2}j + \frac{3}{2}k\right)$ is parallel to **a** and the vector $\left(\frac{1}{2}j + \frac{1}{2}k\right)$ is perpendicular **a**.

### Work

One application of vector projections lies in the computation of the work done by a force.
*If a constant force* **F** *moves an object from point* $P$ *to point* $Q$*, we refer to the vector* $\mathbf{d} = \overrightarrow{PQ}$ *as the displacement vector. The work done is the product of the component of* **F** *along* **d** *and the distance:*

$$\begin{aligned} W &= \text{comp}_{\mathbf{d}}^{\mathbf{F}} \|\mathbf{d}\| \\ &= \frac{\mathbf{F}.\mathbf{d}}{\|\mathbf{d}\|}\|\mathbf{d}\| = \mathbf{F}.\mathbf{d} \end{aligned}$$

**Example 1.18** A force $\mathbf{F} = 2i + 3j + 4k$ moves a particle along the line segment from the point $P(1,2,1)$ to the point $Q(3,6,5)$. Find the work done by the force if $\|\mathbf{F}\|$ is measured in newtons and $\|\mathbf{D}\|$ is measured in meters.

**solution:** The displacement vector is $\|\mathbf{d}\| = \overrightarrow{PQ} = 2i + 4j + 4k$. Therefore, the work done by F is
$W = \mathbf{F}.\mathbf{d} = 32$ joules.

**Example 1.19** A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle of $30^o$ above the horizontal. Find the work done by the force.

**Solution:** $W = \mathbf{F}.\mathbf{d} = \|\mathbf{F}\|\|\mathbf{d}\|\cos\theta = 70)(100)(\frac{\sqrt{3}}{2}) = 3500\sqrt{3} = 6062.18\ J$

**Exercise 1.1**

1. Determine whether the following vectors in $\mathbb{R}^3$ are parallel or not
   (a) $\overrightarrow{A} = 2i - 3j$ and $\overrightarrow{B} = -3i + 2j + k$
   (b) $\overrightarrow{A} = 2i + 4jk$ and $\overrightarrow{B} = -i - 2j + \frac{1}{2}k$
2. Find the value(s) of $x$ such that the vector $\overrightarrow{A} = (1,4,3)$ and $\overrightarrow{B} = (x,-1,2)$ are orthogonal.
3. Find the angle between $\overrightarrow{A} =< -1,-1,1 >$ and $\overrightarrow{B} =< \sqrt{6},1,1 >$
4. Find any unit vectors that are perpendicular to the vector $\overrightarrow{A} =< 3,4 >$
5. If the angle between the vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ is $\theta = \frac{\pi}{6}$ with each other and $\|\overrightarrow{A}\| = \sqrt{3}$ and $\|\overrightarrow{B}\| = 1$, then calculate the cosine the angle between the vectors $\overrightarrow{A} + \overrightarrow{B}$ and $\overrightarrow{A} - \overrightarrow{B}$.
6. Find the projection of $\overrightarrow{A} =< -1,3,1 >$ on to $\overrightarrow{B} =< 2,4,3 >$
7. Suppose $\overrightarrow{a}$ and $\overrightarrow{b}$ are orthogonal, show that $proj_b^a = 0$.
8. Suppose $\overrightarrow{a}$ and $\overrightarrow{b}$ are parallel, show that $proj_a^b = b$.
9. Express the vector $\overrightarrow{b} = 2i + j - 3k$ as the sum of a vector parallel to $\overrightarrow{a} = 3i - j$ and a vector orthogonal to $\overrightarrow{a}$
10. Find the work done by a force $\mathbf{F} = 5k$ in moving an object along the line from the origin to the point $P(1,1,1)$ distance in meters.

## 1.6 The Cross Product

**Definition 1.6.1** Let $\mathbf{a} = a_1 i + a_2 j + a_3 k$ and $\mathbf{b} = b_1 i + b_2 j + b_3 k$ be any two vectors in space. Then the **cross product** of **a** and **b** is the vector

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)i + (a_3b_1 - a_1b_3)j + (a_1b_2 - a_2b_1)k$$

**Example 1.20** Let $\mathbf{a} = <1,3,2>$, and $\mathbf{b} = <2,-1,-1>$. Find $\mathbf{a}\times\mathbf{b}$

**Example 1.21** Show that $\mathbf{a}\times\mathbf{a} = 0$

**Theorem 1.6.1** The vector $\mathbf{a}\times\mathbf{b}$ is orthogonal to both **a** and **b**.

*Proof.* Exercise ■

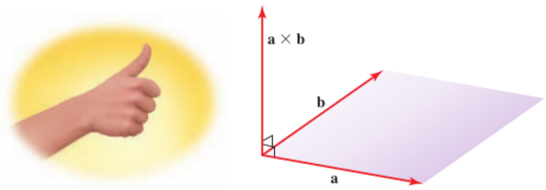



Figure 1.21: The vector $\mathbf{a}\times\mathbf{b}$ is orthogonal to both **a** and **b** with direction determined by the right-hand rule.

**Theorem 1.6.2** Let **a** and **b** be vectors in space. Then

$$\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\|\sin\theta$$

where $\theta$ is the angle between **a** and **b** and $0 \le \theta \le \pi$

**Example 1.22** Let $\mathbf{a} = 2i + 3j$ and $\mathbf{b} = 2j + k$. Find a unit vector **n** that is orthogonal to both **a** and **b**.

**Solution:** A vector that is orthogonal to both **a** and **b** is

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} i & j & k \\ 2 & 3 & 0 \\ 0 & 2 & 1 \end{vmatrix} = 3i - 2j + 4k$$

The length of $\mathbf{a}\times\mathbf{b}$ is

$$\|\mathbf{a}\times\mathbf{b}\| = \sqrt{3^2 + (-2)^2 + 4^2} = \sqrt{29}$$

Therefore, a unit vector that is orthogonal to both **a** and **b** is

$$\mathbf{n} = \frac{\mathbf{a}\times\mathbf{b}}{\|\mathbf{a}\times\mathbf{b}\|} = \frac{3}{\sqrt{29}}i - \frac{2}{\sqrt{29}}j + \frac{4}{\sqrt{29}}k$$

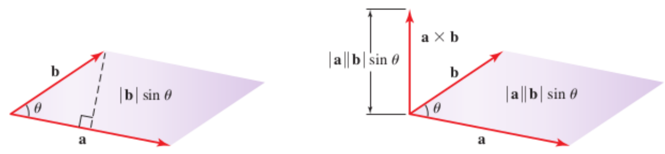
Two nonzero vectors **a** and **b** are parallel if and only if $\mathbf{a}\times\mathbf{b} = 0$

### 1.6.1 Finding the Area of a parallelogram

Consider the parallelogram determined by the vectors **a** and **b** shown in Figure 1.22a. The altitude of the parallelogram is $\|\mathbf{b}\|\sin\theta$ and the length of its base is $\|\mathbf{a}\|$, so its area is

$$A = \|\mathbf{a}\|\|\mathbf{b}\|\sin\theta = \|\mathbf{a}\times\mathbf{b}\|$$

Thus, the length of the cross product $\mathbf{a}\times\mathbf{b}$ and the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$ have the same numerical value. (See Figure 1.22b.)




(a) The parallelogram determined by a and b

(b) The length of a × b is numerically equal to the area of the parallelogram determined by a and b.

Figure 1.22:

R The area of the triangle determined by $\mathbf{a}$ and $\mathbf{b}$ is $\frac{1}{2}\mathbf{a}\times\mathbf{b}$

■ **Example 1.23** Find the area of the triangle with vertices $P(3,-3,0)$, $Q(1,2,2)$, and $R(1,-2,5)$. ■

**Solution:** The area of $\triangle PQR$ is half the area of the parallelogram determined by the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. Now $\overrightarrow{PQ} =< -2,5,2 >$ and $\overrightarrow{PR} =< -2,1,5 >$ so

$$\overrightarrow{PQ}\times\overrightarrow{PR} = 23i+6j+8k$$

Therefore, the area of the parallelogram is

$$\|\overrightarrow{PQ}\times\overrightarrow{PR}\| = \sqrt{23^2+6^2+8^2} = \sqrt{629} \approx 25.1$$

so the area of the required triangle is $\frac{1}{2}\sqrt{629}$ or approximately 12.5.

**Exercise 1.2** Let $\mathbf{a} = 3i+2j-6k$ and $\mathbf{b} = i-2j+2k$. Find the area of the triangle formed by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$. Ans $2\sqrt{17}$ square units. ■

**Properties of the Cross Product**

If $\mathbf{a},\mathbf{b},$ and $c$ are vectors and c is a scalar, then

1. $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$
2. $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$
3. $(\mathbf{a}+\mathbf{b})\times\mathbf{c} = \mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}$
4. $c(\mathbf{a}\times\mathbf{b}) = (c\mathbf{a})\times\mathbf{b} = \mathbf{a}\times(c\mathbf{b})$
5. $\mathbf{a}\times\mathbf{0} = \mathbf{0}\times\mathbf{a} = \mathbf{0}$
6. $\mathbf{a}\times\mathbf{a} = 0$
7. $\mathbf{a}.(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b}).\mathbf{c}$
8. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}.\mathbf{c})\mathbf{b}-(\mathbf{a}.\mathbf{b})\mathbf{c}$

### The Scalar Triple Product

If $\mathbf{a},\mathbf{b},$ and $\mathbf{c}$ are vectors in three-dimensional space. The dot product of $\mathbf{a}$ and $\mathbf{b}\times\mathbf{c}$, $\mathbf{a}.(\mathbf{b}\times\mathbf{c})$, is called the scalar triple product. If we write $\mathbf{a} = a_1 i+a_2 j+a_3 k$, $\mathbf{b} = b_1 i+b_2 j+b_3 k$ and $\mathbf{c} = c_1 i+c_2 j+c_3 k$, then by direct computation,

$$\mathbf{a}.(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The geometric significance of the scalar triple product can be seen by examining the parallelepiped determined by the vectors $\mathbf{a},\mathbf{b},$ and $\mathbf{c}$. (See Figure 1.23.) The base of the parallelepiped is a parallelogram with adjacent sides determined by $\mathbf{b}$, and $\mathbf{c}$ with area $\|\mathbf{b},$ and $\mathbf{c}\|$. If $\theta$ is the angle between $\mathbf{a}$ and $\mathbf{b}\times\mathbf{c}$, then the height of the parallelepiped is given by $h = \|\mathbf{a}\||\cos\theta|$. Therefore, the volume of the parallelepiped is

$$\begin{aligned} V &= \|\mathbf{b}\times\mathbf{c}\|\,|\|\mathbf{a}\|\cos\theta| && \text{(area of base . height)} \\ &= |\mathbf{a}.(\mathbf{b}\times\mathbf{c})| \end{aligned}$$

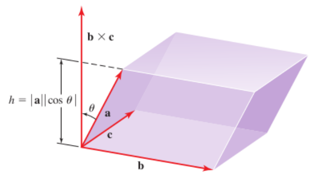



Figure 1.23: The volume V of the parallelepiped is equal to $|\mathbf{a}.(\mathbf{b}\times\mathbf{c})|$

■ **Example 1.24** Find the volume of the parallelepiped with three adjacent edges formed by the vectors $\mathbf{a} = i+2j+3k$, $\mathbf{b} = 4i+5j+6k$ and $\mathbf{c} = 7i+8j$ ■

**Solution:** The volume of the parallelepiped is

$$V = |\mathbf{a}.(\mathbf{b}\times\mathbf{c})| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{vmatrix} = |1(0-48)-2(0-42)+3(32-35)| = 27$$

**Test for Coplanar Vectors**
The vector $\mathbf{a}, \mathbf{b},$ and $\mathbf{c}$ are coplanar if and only if $\mathbf{a}.(\mathbf{b}\times\mathbf{c}) = 0$

■ **Example 1.25** Show that the vectors $\mathbf{a} =< 1,4,-7 >, \mathbf{b} =< 2,-1,4 >,$ and $\mathbf{c} =< 0,-9,18 >$ are coplanar. ■

**Solution:**

$$\mathbf{a}.(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} = |1(-1(18)-(-9)4)-4(2(18)-0)+-7(2(-9)-0)| = 0$$

### Torque

The idea of a cross product occurs often in physics. In particular, we consider a force F acting on a rigid body at a point given by a position vector r . (For instance, if we tighten a bolt by applying a force to a wrench as in Figure 1.24, we produce a turning effect.) The torque (relative to the origin) is defined to be the cross product of the position and force vectors

$$\tau = \mathbf{r}\times\mathbf{F}$$

and measures the tendency of the body to rotate about the origin. The direction of the torque vector indicates the axis of rotation. According to Theorem 1.6.2, the magnitude of the torque vector is

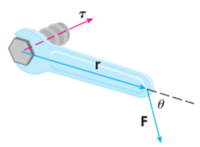



Figure 1.24:

$$\|\tau\| = \|\mathbf{r}\times\mathbf{F}\| = \|\mathbf{r}\|\|\mathbf{F}\|\sin\theta$$

where $\theta$ is the angle between the position and force vectors. Observe that the only component of **F** that can cause a rotation is the one perpendicular to **r**, that is, $\|\mathbf{F}\|\sin\theta$ . The magnitude of the torque is equal to the area of the parallelogram determined by **r** and **F**.

**Example 1.26** A bolt is tightened by applying a 40-N force to a 0.25-m wrench as shown in Figure 1.25. Find the magnitude of the torque about the center of the bolt. ■

**Solution:**

The magnitude of the torque vector is

$$\begin{aligned}\tau &= \mathbf{r}\times\mathbf{F}\\ &= (0.25)(40)\sin 75^o = 10\sin 75^o\end{aligned}$$

If the bolt is right-threaded, then the torque vector itself is

$$\tau = \|\tau\|\mathbf{n}$$

where **n** is a unit vector directed down into the page.

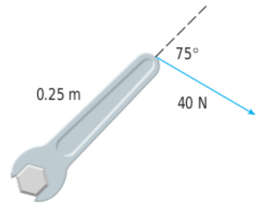



Figure 1.25:

**Exercise 1.3**

1. Find the area of a triangle whose vertices are $A = (1,-1,0)$, $B = (2,1,-1)$, and $C = (-1,1,2)$
2. Find the area of a parallelogram having diagonals $\overrightarrow{a} = 3i + j - 2k$ and $\overrightarrow{b} = i - 3j + 4k$.
3. Find the volume of the parallelepiped with edges $\overrightarrow{u} = i + k$, $\overrightarrow{v} = 2i + j + 4k$ and $j + k$.
4. Find a unit vector perpendicular to the plane of $P(1,-1,0), Q(2,1,-1)$ and $R(-1,1,2)$.
5. Show that the vectors $\mathbf{a} =< 1,5,-2 >, \mathbf{b} =< 3,-1,0 >$, and $\mathbf{c} =< 5,9,-4 >$ are coplanar.

■

## 1.7 Lines and Planes in Space

**Definition 1.7.1** A vector $\mathbf{v} =< a,\ b,\ c >$ is said to be parallel to a line $L$ if $\mathbf{v}$ is parallel to $\overrightarrow{P_0P}$ for any two points $p_0$ and $p_1$ on $L$.

A line $L$ in $\mathbb{R}^3$ is determined by a given point $p_0(x_0, y_0, z_0)$ on $L$ and a parallel vector $\mathbf{v} =< a,\ b,\ c >$ (directional vector) to $L$.

### 1.7.1 Equation of a line in space

Suppose that the line $L$ passes through the point $P_0(x_0, y_0, z_0)$ and has the same direction as the vector $\mathbf{v} =< a,\ b,\ c >$.

Let $P(x,y,z)$ be any point on $L$. Then the vector $\overrightarrow{P_0P}$ is parallel to $v$. But two vectors are parallel if and only if one is a scalar multiple of the other. Therefore, there exists some number $t$ , called a parameter, such that

$$\overrightarrow{P_0P} = t\mathbf{v}$$

or since, $\overrightarrow{P_0P} =< x-x_0,\ y-y_0,\ z-z_0 >= t < a,\ b,\ c >=< ta,\ tb,\ tc >$ we have

$$< x-x_0,\ y-y_0,\ z-z_0 >$$

Equating the corresponding components of the two vectors then yields

$$x-x_0 = ta,\ y-y_0 = tb,\ z-z_0 = tc$$

Solving these equations for $x, y,$ and $z$ , respectively, gives the following standard parametric equations of the line $L$.

**Definition 1.7.2 — Parametric Equations of a Line.** The parametric equations of the line passing through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\mathbf{v} =< a,\ b,\ c >$ are

$$< x = x_0 + ta,\ y = y_0 + tb,\ z = z_0 + tc >$$

■ **Example 1.27** Find parametric equations for the line passing through the point $P_0(2,1,3)$ and parallel to the vector $\mathbf{v} =< 1,2,2 >$. ■

**Solution:** $x = -2 + t,\ y = 1 + 2t,$ and $z = 3 - 2t$

**Definition 1.7.3 — Symmetric Equations of a Line.** The symmetric equations of the line $L$ passing through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

R Suppose $a = 0$ and both $b$ and $c$ are not equal to zero, then the parametric equations of the line take the form

$$x = x_0,\ y = y_0 + tb,\ z = z_0 + tc$$

and the line lies in the plane $x = x_0$ (parallel to the yz -plane). Solving the second and third equations for $t$ leads to

$$x = x_0,\ \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

which are the symmetric equations of the line.

■ **Example 1.28**
(a) Find parametric equations and symmetric equations for the line L passing through the points $P(-3,3,-2)$ and $Q(2,-1,4)$ .
(b) At what point does L intersect the xy -plane?
■

**Solution:**
(a) The direction of $L$ is the same as that of the vector $\overrightarrow{PQ} =< 5,-4,6 >$. Since $L$ passes through $P(-3,3,-2)$, the parametric equations of the line is

$$x = -3 + 5t,\ \ y = 3 - 4t,\ \ \text{and } z = -2 + 6t$$

The symmetric equations for $L$ is:

$$\frac{x+3}{5} = \frac{y-3}{-4} = \frac{z+2}{6}$$

(b) At the point where the line intersects the xy-plane, we have $z = 0$. So setting $z = 0$ in the third parametric equation, we obtain $t = \frac{1}{3}$. Substituting this value of $t$ into the other parametric equations gives the required point as $\left(-\frac{4}{3}, \frac{5}{3}, 0\right)$.

■ **Example 1.29** Find the parametric equations of the line that pass through the point $(-1,3,-2)$ and is perpendicular to the vectors $\vec{A} = 3i + 4j + k$ and $\vec{B} = i + 2j$ ■

**Solution:** Since the line is perpendicular to the vectors $\vec{A}$ and $\vec{B}$, it is parallel to the vector product of $\vec{A}$ and $\vec{B}$. Since

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} i & j & k \\ 3 & 4 & 1 \\ 1 & 2 & 0 \end{vmatrix} = -2i + j + 2k$$

it follows that the parametric equations of the line are

$$x = -1 - 2t,\ y = 3 + t,\ z = -2 + 2t$$

**Definition 1.7.4** Let $l_1$ and $l_2$ be two lines in $\mathbb{R}^3$, with parallel vectors **a** and **b**, respectively, and let $\theta$ be the angle between **a** and **b**.

(a) The lines $l_1$ and $l_2$ are **parallel** whenever **a** and **b** are parallel.
(b) If $l_1$ and $l_2$ intersect, then
  i. the angle between $l_1$ and $l_2$ is $\theta$ and
  ii. the lines $l_1$ and $l_2$ are orthogonal whenever **a** and **b** are orthogonal.

**Example 1.30** Let $L_1$ be the line with parametric equations

$$x = 1 + 2t,\ y = 2 - 3t,\ \text{ and } \ z = 2 + t$$

and $L_2$ be the line with parametric equations

$$x = 3 - 4t,\ y = 1 + 4t,\ \text{ and } \ z = -3 + 4t$$

(a) Show that the lines $L_1$ and $L_2$ are not parallel to each other.
(b) Do the lines $L_1$ and $L_2$ intersect? If so, find their point of intersection.

**Solution:**

(a) $L_1$ has the same direction as the vector $\vec{v_1} =< 2, -3, 1 >$. Similarly, $L_2$ has direction given by the vector $\vec{v_2} =< -4, 4, 4 >= 4 < -1, 1, 1 >$. Since $\vec{v_1}$ is not a scalar multiple of $\vec{v_2}$, the vectors are not parallel, so $L_1$ and $L_2$ are not parallel as well.

(b) Suppose that $L_1$ and $L_2$ intersect at the point $P_0(x_0, y_0, z_0)$ . Then there must exist parameter values $t_1$ and $t_2$ such that

$$x_0 = 1 + 2t_1,\ y_0 = 2 - 3t_1,\ \text{ and } \ z_0 = 2 + t_1$$

and

$$x_0 = 3 - 4t_2,\ y_0 = 1 + 4t_2,\ \text{ and } \ z_0 = -3 + 4t_2$$

This leads to the system of three linear equations

$$\begin{aligned} 1 + 2t_1 &= 3 - 4t_2 \\ 2 - 3t_1 &= 1 + 4t_2 \\ 2 + t_1 &= -3 + 4t_2 \end{aligned}$$

Adding the first two equations gives $t_1 = -1$. Substituting this value of $t_1$ into either the first or the second equation then gives $t_2 = 1$. Finally, substituting these values of $t_1$ and $t_2$ into the third equation gives $2 - 1 = -3 + 4(1) = 1$ , which shows that the third equation is also satisfied by these values. We conclude that $L_1$ and $L_2$ do indeed intersect at a point. To find the point of intersection, substitute $t_1 = -1$ into the parametric equations defining $L_1$ , or, equivalently substitute $t_2 = 1$ into the parametric equations defining $L_2$. In both cases we find that $x_0 = -1, y_0 = 5,$ and $z_0 = 1$ , so the point of intersection is $(-1, 5, 1)$ .

**Definition 1.7.5** Two lines in space are said to be **skew** if they do not intersect and are not parallel.

**Example 1.31** As two planes fly by each other, their flight paths are given by the straight lines

$$L_1 :\ x = 1 - t,\ y = -2 - 3t,\ \text{ and } \ z = 4 + t$$

and

$$x = 2 - 2t,\ y = -4 + 3t,\ \text{ and } \ z = 1 + 4t$$

Show that the lines are skew and, therefore, that there is no danger of the planes colliding.

**Solution:** The directions of $L_1$ and $L_2$ are given by the directions of the vectors $v_1 =< -1, -3, 1 >$ and $v_2 =< -2, 3, 4 >$, respectively. Since one vector is not a scalar multiple of the other, the lines $L_1$ and $L_2$ are not parallel. Next, suppose that the two lines do intersect at some point $P_0(x_0, y_0, z_0)$ . Then

$$x_0 = 1 - t_1,\ y_0 = -2 - 3t_1,\ \text{ and } \ z_0 = 4 + t_1$$

and

$$x_0 = 2 - 2t_2,\ y_0 = -4 + 3t_2, \text{ and } z_0 = 1 + 4t_2$$

for some $t_1$ and $t_2$. Equating the values of $x_0, y_0,$ and $z_0$ then gives

$$\begin{aligned} 1 - t_1 &= 2 - 2t_2 \\ -2 - 3t_1 &= -4 + 3t_2 \\ 4 + t_1 &= 1 + 4t_2 \end{aligned}$$

Solving the first two equations for $t_1$ and $t_2$ yields $t_1 = \frac{1}{9}$ and $t_2 = \frac{5}{9}$ substituting these values of $t_1$ and $t_2$ into the third equation gives $\frac{37}{9} = \frac{29}{9}$ a contradiction. This shows that there are no values of $t_1$ and $t_2$ that satisfy the three equations simultaneously. Thus, $L_1$ and $L_2$ do not intersect. We have shown that $L_1$ and $L_2$ are skew lines, so there is no possibility of the planes colliding.

### Distance from a point to a Line

Let $d$ represent the distance from the point $Q$ to the line through the points $P$ and $R$. We have

$$D = \|\overrightarrow{PQ}\| \sin\theta$$

where $\theta$ is the angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. Thus,

$$\|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \|\overrightarrow{PQ}\|\|\overrightarrow{PR}\| \sin\theta = \|\overrightarrow{PR}\|(D)$$

Solving this for $D$, we get

$$D = \frac{\|\overrightarrow{PQ} \times \overrightarrow{PR}\|}{\|\overrightarrow{PR}\|}$$

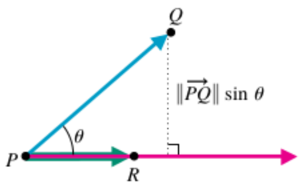



Figure 1.26:

■ **Example 1.32** Find the distance from the point $P(1,-2,3)$ to the line with parametric equation $x = 3 + t,\ y = -1 - 2t,\ z = 4 + t$. ■

**Solution:** Choose any point from the line, say, $Q(3,-1,4)$. Then $\overrightarrow{PQ} = <2,1,1>$ and the direction vector is $\mathbf{v} = <1,-2,1>$, hence

$$\overrightarrow{PQ} \times \mathbf{v} = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ 1 & -2 & 1 \end{vmatrix} = 3i - j - 5k. \text{ Thus, the distance is}$$

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{v}\|}{\|\mathbf{v}\|} = \frac{\sqrt{3^2 + (-1)^2 + (-5)^2}}{\sqrt{1^2 + (-2)^2 + 1^2}} = \frac{\sqrt{35}}{\sqrt{6}}$$

■ **Example 1.33** Find the distance $D$ between the two parallel lines $L_1$ and $L-2$ where

$$L_1 : \frac{x-1}{2} = \frac{y+1}{-2} = \frac{z-2}{1}, \quad L_2 : x = 2 + t, y = -t, z = 3 + \frac{1}{2}t$$ ■

**Solution:** $P = (1,-1,2)$ and $Q = (2,0,3) \implies \overrightarrow{PQ} = <2,0,3>$ and $\mathbf{v} = <2,-2,1>$. Thus

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{v}\|}{\|\mathbf{v}\|} = \frac{\sqrt{26}}{3}$$

### 1.7.2 Equations of Planes

A plane in space is uniquely determined by specifying a point $P_0(x_0,y_0,z_0)$ lying in the plane and a vector $\mathbf{n} =< \mathbf{a},\mathbf{b},\mathbf{c} >$ that is normal (perpendicular) to it.
To find an equation of the plane, let $P(x,y,z)$ be any point in the plane. Then the vector $\overrightarrow{P_0P}$ must be orthogonal to $\mathbf{n}$. But two vectors are orthogonal if and only if their dot product is equal to zero. Therefore, we must have

$$\mathbf{n}.\overrightarrow{P_0P} = 0$$

Since $\overrightarrow{P_0P} =< x-x_0,\ y-y_0,\ z-z_0 >$, we have

$$< a,b,c > . < x-x_0,\ y-y_0,\ z-z_0 >= 0$$
$$\Longrightarrow \quad a(x-x_0)+b(y-y_0)+c(z-z_0) = 0$$

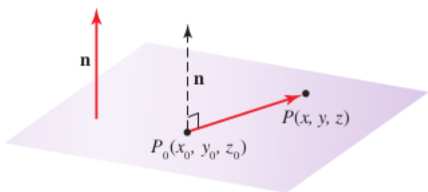



Figure 1.27:

**Definition 1.7.6** The standard form of the equation of a plane containing the point $P_0(x_0,y_0,z_0)$ and having the normal vector $\mathbf{n} =< \mathbf{a},\mathbf{b},\mathbf{c} >$ is

$$a(x-x_0)+b(y-y_0)+c(z-z_0) = 0$$

■ **Example 1.34** Find an equation of the plane containing the point $P_0(1,2,3)$ and having a normal vector, $< 4,5,6 >$ and sketch the plane. ■

**Solution:** The equation of the plane is

$$a(x-x_0)+b(y-y_0)+c(z-z_0) = 0 \Longrightarrow 4(x-1)+5(y-2)+6(z-3) = 0$$
$$\Longrightarrow 4x+5y+6z = 32$$

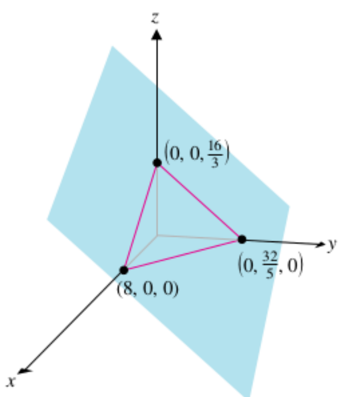



Figure 1.28: The plane through $(8,0,0)$, $\left(0,\frac{32}{5},0\right)$, $\left(0,0,\frac{16}{3}\right)$

■ **Example 1.35** Find an equation of the plane that passes through the points $P(1,3,2)$, $Q(3,-1,6)$ and $R(5,2,0)$. ■

**Solution:** The vectors $\overrightarrow{PQ} =< 2,-4,4 >$ and $\overrightarrow{PR} =< 4,-1,-2 >$ lie in the plane, so the vector $\|\overrightarrow{PQ} \times \overrightarrow{PR}\|$ is normal to the plane. Denoting this vector by **n**, we have

$$\mathbf{n} = \|\overrightarrow{PQ} \times \overrightarrow{PR} = \| \begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12i + 20j + 14k$$

Hence, using the point $P(1,3,2)$ in the plane (any of the other two points will also do) and the normal vector **n** just found, with $a = 12, b = 20, c = 14, x_0 = 1, y_0 = 3,$ and $z_0 = 2$, we obtain the equation of the desired plane,

$$a(x-x_0)+b(y-y_0)+c(z-z_0) = 0 \implies 12(x-1)+20(y-3)+14(z-2) = 0$$
$$\implies 12x+20y+14z = 100 \implies 6x+10y+7z = 50$$

### Parallel and Orthogonal Planes

Two planes with normal vectors **m** and **n** are parallel to each other if m and n are parallel; the planes are orthogonal if **m** and **n** are orthogonal.

■ **Example 1.36** Find an equation of the plane containing $P(2,-1,3)$ and parallel to the plane defined by $2x+3y+4z = 6$ . ■

**Solution:** The normal vector of the given plane is $\mathbf{n} =< 2,-3,4 >$. Since the required plane is parallel to the given plane, it also has **n** as a normal vector. Hence, the equation of the plane is

$$2(x-2)+(-3)(y-(-1))+4(z-3) = 0 \implies 2x-3y+4z = 19$$

### The Angle Between Two Planes

Two distinct planes in space are either parallel to each other or intersect in a straight line. If they do intersect, then the angle between the two planes is defined to be the acute angle between their normal vectors

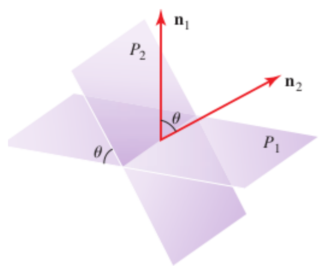



Figure 1.29: The angle between two planes is the angle between their normal vectors.

■ **Example 1.37** Find the angle between the two planes defined by $3x-y+2z = 1$ and $2x+3y-z = 4$. ■

**Solution:** The normal vectors of these planes are

$$\mathbf{n_1} =< 3,-1,2 > \text{ and } \mathbf{n_2} =< 2,3,-1 >$$

Therefore, the angle $\theta$ between the planes is given by

$$\cos\theta = \frac{\mathbf{n_1}.\mathbf{n_2}}{\|\mathbf{n_1}\|\|\mathbf{n_2}\|} = \frac{1}{14} \implies \theta = \cos^{-1}\left(\frac{1}{14}\right) \approx 86^o$$

■ **Example 1.38** Find parametric equations for the line of intersection of the planes defined by $3x - y + 2z = 1$ and $2x + 3y - z = 4$. ■

**Solution:**

### The Distance Between a Point and a Plane

Let $\pi$ be a plane with normal $\mathbf{n}$ and let $P_1$ be a point not on the plane. Then the distance $D$ between $P_1$ and the plane $\pi$ is given by

$$D = \frac{|\mathbf{n}.\overrightarrow{P_1P_0}|}{\|\mathbf{n}\|}$$

where $P_0$ is any point on the plane.

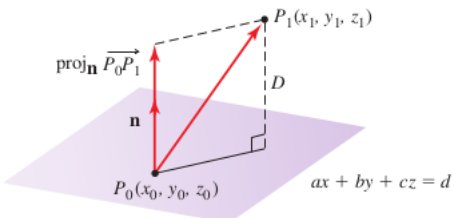



Figure 1.30:

■ **Example 1.39** Find the distance between the point $(-2, 1, 3)$ and the plane $2x - 3y + z = 1$ . ■

**Solution:** $D = \dfrac{5}{\sqrt{14}}$

**Exercise 1.4**

1. Find parametric and symmetric equations of the line passing through the point $(1, 2, -1)$ and parallel to the line with parametric equations $x = -1 + t, y = 2 + 2t$ , and $z = -2 - 3t$ . At what points does the line intersect the coordinate planes?
2. Find symmetric equation of a line $L$ containing the point $P_0(0, 1, -1)$ and perpendicular to the line $L$ with parametric equation
$$x = 2 + 3t,\ y = -2 - 2t,\ z = 4t$$
3. Find parametric equations of the line that is parallel to the line with equation
$$\frac{x-1}{4} = \frac{y+4}{5} = \frac{z+1}{2}$$
and contains the point of intersection of the lines
$$L_1: \ x = 4 + t,\ y = 5 + t,\ z = -1 + 2t$$
$$L_2: \ x = 6 + 2t,\ y = 11 + 4t,\ z = - + t$$
4. Find the equation of a plane containing the point $P_0(1, -1, 2)$ and having normal $\mathbf{n} = \frac{1}{2} i + 2j - k$.
5. Find the equation of a plane containing the point $P(1, -1, 1), Q(2, 3, 0)$ and $R(-1, 2, -2)$.
6. Find the distance between the point $P_1(-4, 2, 7)$ and the plane $2x - 3y + 4z = 1$

■